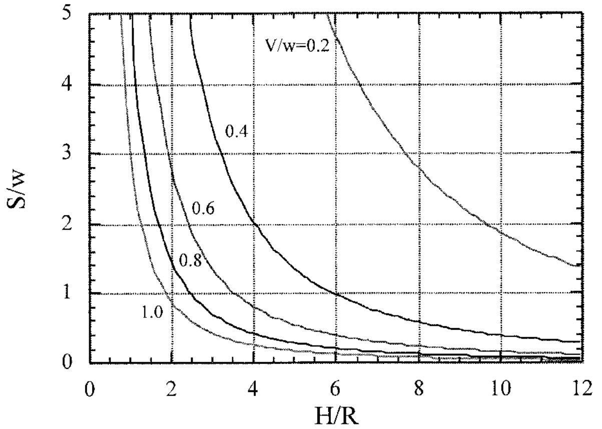

Fig. 1 Maximum induced surface velocity contours.

speed of 32.4 m/s (S = 3.0) and a drop height of 48.0 m (H = 6.0) will generate an estimated maximum induced surface velocity of approximately 2.6 m/s (V = 0.24). The utility of these results is then obvious and may be used to infer the potential for any helicopter to promote the sideways spread of fire, pesticides, dusts, aerosols, or other contaminants.

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Neglect of Wake Roll-Up in Theodorsen's Theory of Propellers

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Introduction

THIS note addresses a paper by Schouten,[1] concerning the optimum propeller efficiency obtainable, neglecting profile drag, under specified operating conditions.[2–4] Schouten[1] directs criticism at the well-known theory of Theodorsen,[4–6] which models the propeller wake as a rigid backward moving (multiple) helicoid vortex sheet. He argues that the theory entails substantial error by failing to allow for the vortex sheet roll-up that occurs in reality. The roll-up, Schouten says, ensures that the static pressure in the wake tends toward ambient[7]: that it eliminates an unrealistic pressure rise permitted by the helicoidal scenario. This in turn is claimed to lead to an underestimation of the required power. As a corollary, the predictions of this ideal efficiency are judged to be substantially too high.

We find the arguments[1,7] concerning wake static pressure to be only partially correct and the conclusions concerning propeller power and efficiency untenable. Although the arguments are persuasive, the numerical comparisons, implicitly equating apples and oranges, are not valid. We conclude that neglect of wake roll-up only slightly mispredicts the efficiency. The basis for these remarks is developed further in the following text.

The underlying scenario goes back to Betz,[2] for lightly loaded propellers. Betz showed that this ideal efficiency is associated with an optimum loading that yields a special trailing vortex pattern: one whose induced velocity is such that it behaves like a (multiple) rigid helicoid sheet or screw surface moving axially backward. Goldstein[3] developed the analytic theory, leading to practical, accurate, performance prediction. Theodorsen[4] generalized this to apply, allowing for wake contraction, to heavily loaded propellers. He presented rigorous proof that this rigid helicoid wake corresponds to maximum efficiency for his flow model.

The nonrigid behavior of the wake is a matter that Theodorsen[4] treats rather dismissively: ". . . the theory [he says] . . . may to some extent be 'overidealized'. The vortex surface is in fact unstable and will therefore not maintain its ideal shape for any length of time." Both Theodorsen[4] and Schouten[1] are referring to the rolling up of the helicoid vortex sheet. This is a phenomenon that has been ignored in the theory, and Schouten deserves credit for raising these matters with a penetrating analysis.

Issue must be taken with Schouten's arguments and with the numerical results flowing therefrom: these results are based on inappropriate values of κ and ε inserted in a somewhat flawed equation. Thus it is invalid to use these numbers (apples vs oranges) in comparisons with Theodorsen's theory[4] that neglect the vortex rolling up. In particular, Schouten's[1] conclusion that the theory substantially overpredicts the ideal efficiency is not supported.

Is the Helicoid Sheet Force-Free?

We do agree, however, with Schouten's assertion[1] that wake edge forces are required to suppress roll-up. It is only with the implications that we disagree. Quoting Betz[8] for the analogous planar case,

> this motion would only be possible for any length of time if the area of discontinuity [here the helicoid] actually were rigid. By flowing around the edges, laterally directed suction forces P occur, which only could be taken up by a rigid plate [helicoid]. These forces are absent when the area of discontinuity is other than rigid, as a result of which the suction P effects other motions: starting at the edges, it unrolls and gradually forms two distinct vortices.

Theodorsen's postulation of no roll-up is thus an imposition of rigidity: as a consequence, it implies edge forces as claimed by Schouten.[1] The edge forces on opposite sides of a cylindrical helicoid are directed radially and are opposed: they affect neither the thrust nor the power. The equations for these, given in the following text, are therefore unaffected.

But this casual view overlooked the contraction region in which the edge forces are tilted: backward and circumferen-

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tially. Acting on the implied rigid wake, these would make spurious contributions to the propeller forces: the thrust would be underestimated and the power overestimated. This is because they are implicitly included in the momentum and energy balance equations. Thus, by omission of edge forces, the Theodorsen theory[1] underestimates the efficiency. Noting that the helicoid contraction is but a few percent at most, it would be expected that these effects should be small. Some laborious estimates support this conclusion, yielding underestimates of efficiency of under 1 percent for a wide range of circumstances. A more detailed examination of edge force effects is under way.

General Equations for Thrust and Power

As noted in Refs. 4–6, Theodorsen's basic equations effectively state the steady flow conservation laws for a large cylindrical control surface enveloping the propeller and its wake; this is coaxial with the spin axis, taken along z, measured from the hub.

Conservation of momentum and mass lead to thrust as

$$T = \int_{-\infty}^{\infty} [(p - p_0) + \rho(V + v_z)v_z] \, dS \quad (1)$$

and conservation of energy and mass lead to power absorbed as

$$P = \int_{-\infty}^{\infty} [(p - p_0) + \rho(v^2/2 + Vv_z)](V + v_z) \, dS \quad (2)$$

where the integrals are taken over an infinite plane (the Trefftz plane) far back ($z \to \infty$). Herein V is stream speed, parallel to z, through the propeller plane, v is the local disturbance velocity, and v_z is its axial component.

Pressure Rise in the Wake: Equations

The pressure rise $(p - p_0)$ in the wake, as $z \to \infty$, is evaluated for convenience from the Bernoulli equation referred to a reference frame fixed to the unperturbed fluid; in this frame it is unsteady:

$$(p - p_0) = -\rho \frac{\partial \phi}{\partial t} - \rho \frac{v^2}{2} \quad (3)$$

Schouten[1] argued that this expression as evaluated from the Theodorsen theory[4] is too high, owing to neglect of the rolling up, and this leads to predicted efficiencies that are too high. We will return to this point later. For the evaluation, as noted by Schouten,[1,7] the flow pattern potential ϕ moves backward through to fluid with some velocity W that differs in the two cases. It follows that the time and space derivatives of ϕ are related by

$$\frac{\partial \phi}{\partial t} = -W \frac{\partial \phi}{\partial x} = -Wv_z \quad (4)$$

For the Theodorsen[4] (helicoid) scenario, W is just the axial velocity w of the rigid helicoid relative to the fluid at rest at infinity. For the Schouten (rolled up) scenario,[1] W must be the local velocity in which a small segment of the rolled up helical vortex lies, induced by the remainder of the vortex system. This is taken by Schouten to be $w/2$ axially. This is generalized to be $w'/2$ to allow for the difference in the respective scenarios. The pressure rise in the wake [Eq. (3)] then takes the following alternate forms:

$$(p - p_0) = \rho v_z w - \rho v^2/2 \quad \text{helicoid wake} \quad (5)$$

$$(p - p_0) = \rho v_z w'/2 - \rho v^2/2 \quad \text{rolled-up wake} \quad (6)$$

Because the flow patterns are in general very different, Schouten's[1] implicit identification of w and w' is questionable. [In the following text, primes will be added to the symbols of Eq. (6).]

Nondimensional Forms

The integrals arising in Eqs. (1) and (2), after the insertion of Eqs. (5) or (6) to eliminate $(p - p_0)$, are expressed in nondimensional form via the definitions

$$\kappa = \int_{-\infty}^{\infty} \frac{v_z \, dS}{wA}, \qquad \mu = \int_{-\infty}^{\infty} \frac{v^2 \, dS}{w^2 A}, \qquad \varepsilon = \int_{-\infty}^{\infty} \frac{v_z^2 \, dS}{w^2 A} \quad (7)$$

when applied to the rigid helicoid velocity field. For the rolled up wake, with its altered velocity field, primes are added to all the symbols including $A = \pi D_\infty^2/4$. (The diameter D_∞ of the helicoid wake far back is often approximated as the propeller diameter D, because the contraction $D \to D_\infty$ as calculated on the helicoid model is small; to this approximation, $A = A'$.)

Using relations (7), Eqs. (1) and (2) may be rewritten in nondimensional form as the respective thrust and power coefficients, writing w/V as $\bar{w}$.

For Theodorsen's scenario (rigid helicoid wake):

$$c_{S,T} = 2T_T/\rho AV^2 = 2\kappa\bar{w}(1 + \bar{w}\varepsilon/\kappa) + \boldsymbol{\bar{w}^2(2\kappa - \mu)} \quad (8)$$

$$c_{P,T} = 2P_T/\rho AV^3 = 2\kappa\bar{w}(1 + \bar{w}\varepsilon/\kappa)(1 + \bar{w}) \quad (9)$$

and for Schouten's scenario (rolled up wake):

$$c_{S,\mathrm{Sch}} = 2T_{\mathrm{Sch}}/\rho AV^2 = 2\kappa'\bar{w}'(1 + \bar{w}'\varepsilon'/\kappa') + \boldsymbol{\bar{w}'^2(\kappa' - \mu')} \quad (10)$$

$$c_{P,\mathrm{Sch}} = 2P_{\mathrm{Sch}}/\rho AV^3 = 2\kappa'\bar{w}'(1 + \bar{w}'\varepsilon'/\kappa')(1 + \bar{w}'/2) \quad (11)$$

In c_S, the terms in boldface correspond to the integrals of $(p - p_0)$ over the Trefftz plane. [In c_P, $(p - p_0)$ is involved as a weighted integral.]

It is seen that roll-up changes the form of the equations: the boldface terms differ, and the factors in c_P are altered; moreover, the addition of primes signals changes in the magnitudes of the variables. Thus, the propeller efficiency $\eta = c_s/c_p$ will appear to differ if we use the same values of $\bar{w}$, κ, μ, and ε for the helicoid wake case of Theodorsen [Eqs. (8) and (9)] and the rolled up wake case of Schouten[1] [Eqs. (10) and (11)]. This is what Schouten has done in his examples, but it amounts to equating apples to oranges. This is because the flows governing these variables are, in general, very dissimilar. The comparison is not quantitatively meaningful, and no inference can be drawn from the numbers. In particular, the conclusion therefrom that wake roll-up substantially reduces propeller efficiency is without foundation.

Returning to the power coefficient c_P, Schouten[1] argues that Theodorsen's required power[4] is smaller than the value dictated by roll-up; presumably, this is at the same V/nD and efficiency. Curiously, the curves of Schouten's Fig. 2 (Ref. 1) show just the reverse. Supporting this, the equations show that Theodorsen's form [Eq. (9)], is actually larger than the form [Eq. (11)] associated with roll-up, for a specific $\bar{w}$ (which is set equal to $\bar{w}'$), etc. This is because Eq. (11) has a factor $(1 + \bar{w})$ in place of the factor $(1 + \bar{w}'/2)$; this is a contribution of $(p - p_0)$. In short, the Theodorsen formulation does not yield an "underestimated required power."

Recall that κ and μ as defined in Eqs. (7), when unprimed, refer to the flow induced by the rigid helicoid motion: Theodorsen has shown that for this special flow the two are equal: $\mu = \kappa$. Thus, μ is always replaced by κ in Theodorsen's equations,[4] and this should be done in the application of Eq. (8): the boldface term is then positive, implying a positive overpressure $(p - p_0)$ in the wake. But Schouten[1] has inferred a similar equality $\mu' = \kappa'$ (along with $\kappa' = \kappa$) for his markedly different flow, designated with primes. The boldface term in Schouten's Eq. (10) is then zero, implying, as he noted, ambient pressure in the wake. (This truncated form is the flawed equation referred to in the Introduction.) However, swirl can reduce the interior pressure of the wake below ambient, so that μ' may be greater than κ'.

Pressure-Velocity Tradeoff

A referee has defended Schouten's[1] equating of κ' and κ, etc., on the grounds that wake kinetic energy must be conserved during roll-up. But nowhere in Eq. (2), expressing conservation of energy and mass, is it required that kinetic energy be conserved. Examination of the equation shows a tradeoff between pressure work performed on the control surface and kinetic energy. Because roll-up, as reported by Schouten, alters the pressure in the wake, so must it alter the kinetic energy. With reduction in $(p - p_0)$ the velocity parameters $\bar{w}$, κ, μ, and ε defined in Eq. (7) are expected to change (to different values designated with primes) so that the equations predict little alteration in efficiency. This follows from an earlier judgement herein (shared by Schouten[1]) that rolling up of the wake only slightly affects the velocities it induces over the propeller blades. Thus, the thrust and torque, hence efficiency, of the Theodorsen-designed propeller[4] should be only slightly modified by roll-up.

Limitations of the Roll-Up Model

The theoretical considerations of roll-up ignore the influence of fluid viscosity and turbulence in the propeller wake. The diffusive behavior of these, especially the turbulence, will further modify the flowfield. Here, again, we judge that the modification will have but minor induction effects at the propeller; hence, minor effects on the predicted performance.

Experimental Evidence

The discussion has been limited to idealized, friction-free, propellers. This applies to the cited expressions of Theodorsen[4] for thrust, power, and efficiency. Allowance for the profile drag of real propellers is made in his section on propeller design[4,6,9]; this uses charts of C_D vs C_L data for two-dimensional blade sections. It involves the assumed helicoid wake-induced displacement velocity (approximately $1/2w$) at the propeller plane, which in turn depends of the specified c_p $(=2P/\rho AV^3)$ plus $\kappa(\lambda_T)$ and $\varepsilon(\lambda_T)$. Quoting Crigler,[9] "For single-rotating propellers, the method yields the same results as the conventional vortex theory with the Goldstein tip corrections applied"; it should approximate the optimum blade loading for maximum efficiency. This equivalent method had been used earlier by Crigler[10] to predict the performance of a series of four-, six-, and eight-blade propellers. Crigler's comparisons with results of prior experimental tests showed remarkably close agreement with the measured thrust and power; hence, efficiency.

The curves of Refs. 5 and 6 yield Theodorsen's ideal efficiency[4]: the value obtainable in the absence of blade friction drag (and for optimum blade loading). We select the four-blade propeller of Fig. 9 of Ref. 10, with blade angle 45 deg, and $V/nD = 2$; C_p $(=P/n^3D^5)$ is near 0.314. Figure 40 of Ref. 6 predicts an ideal efficiency of 92.0%, to be compared with the measured efficiency of 86.0%. A drop of 6% attributable to friction drag is entirely reasonable according to Crigler's calculations.[9]

In summary, a detailed set of calculations[10] by an equivalent to Theodorsen's design method,[4] allowing for friction drag, yields efficiencies (as well as thrust and power) close to measurement. And a prediction on the Theodorsen theory of the inviscid efficiency[1] in an example is quite compatible. These propellers have values of c_p at peak efficiency of order 0.23–0.29; this corresponds to w/V (ratio of wake helicoid axial velocity to flight velocity) of order 0.17 to as high as 0.24.

Higher values of w/V would be expected to incur greater wake roll-up effects. An example is afforded by the eight-blade SR-3 Prop-Fan analyzed by Hanson.[11] He developed and applied a version of compressible lifting surface theory to the prediction of performance. In common with Theodorsen's approach,[4] a rigidly moving helicoidal wake is assumed. In the specified example ($M = 0.8$, $V/nD = 3.002$), the measured C_p is 1.385 at an efficiency of 82.3%. For essentially the same power, Hanson's inviscid lifting surface theory[11] predicts 85.3% and Theodorsen's theory[4] predicts 85.8% (from Fig. 43 of Ref. 6). The excess of 2.5–3% over measurement is quite appropriate: it is an increment caused by friction drag, which is not included in the theory. Thus, the Theodorsen[6] and Hanson[11] predictions of the ideal (inviscid) efficiency have considerable support from this experiment. On the other hand, the graphs of Schouten's[1] Fig. 2 predict 74.5% as the ideal efficiency. This is 7.8% lower than the measured efficiency. In short, the reduction predicted in Schouten's reformulation grossly contradicts experiment.

For Hanson's scenario[11] of the eight-blade propeller, w/V can be determined as 0.47 from Fig. 61 of Ref. 6. These successful predictions appear to show that wake roll-up is a negligible factor for propellers of loading, in one case at least, sufficient to bring w/V up to nearly half; thus it would seem that wake roll-up would not normally be a concern. However, to put the matter to rest and to round out our understanding of propeller behavior, it would be worthwhile to study wake roll-up effects in detail and their effect on propeller performance. This would entail the extension of current CFD studies to allow fully for the upstream induction effects of the extended, rolled up, wake.

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